Update on the Light-Front Quark Model and Mass Spectrum Calculations for Ground State Pseudoscalar and Vector Mesons

Ho-Meoyng Choi,[1] Chueng-Ryong Ji,[2] and Ziyue Li[2]

[1] *Department of Physics, Teachers College, Kyungpook National University, Daegu, Korea 702-701*
[2] *Department of Physics, North Carolina State University, Raleigh, North Carolina 27695-8202*

We present an update on the meson mass spectra calculation with the light-front quark model constrained by the variational principle for the QCD-motivated effective Hamiltonian. By smearing out the Dirac delta function in the hyperfine interaction, we were able to avoid the negative infinity that one encounters when using variational principle for the entire Hamiltonian. We obtained a better fitting for the mass spectra of ground state pseudoscalar and vector mesons from π to Υ, compared to the previous calculation that handled the hyperfine interaction as a perturbation rather than including it in the parameterization process. Our new parameters are further tested in decay constant calculation. We showed that by taking a larger harmonic oscillator basis in our trial wave function, the decay constants calculated from our model can be improved to agree reasonably well with the experimental data.

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I. INTRODUCTION

Effective degrees of freedom to describe a strongly interacting system of hadrons have been one of the key issues in understanding the non-perturbative nature of QCD in the low energy regime. Within an impressive array of effective theories available nowadays, the constituent quark model has been quite useful in providing a good physical picture of hadrons just like the atomic model for the system of atoms. Absorbing the complicated sea quark pairs and gluonic interactions into the masses of constituent quarks and effective interactions in the Hamiltonian, one can make the problem much simpler yet still keep some key features of the underlying QCD to provide useful predictions [1]. The commonly used valence constituent quark models truncate the Fock space to the lowest sector and simplify the whole system to a bound state of dressed valence quark/antiquark constituents with QCD inspired effective potentials. Although a clear link between the QCD and this type of constituent quark models is still pending, the justification of effective potentials are usually provided by the flux tube configurations generated by the gluon fields as well as the one-gluon-exchange calculation in QCD [2, 3].

In addition, relativistic effects play an important role in describing the low-lying hadrons made of u, d, and s quarks and antiquarks. Thus, a proper way of dealing with the relativistic aspect of the hadron system is also quite essential. The formulation of light-front dynamics(LFD) [4], or the front form dynamics introduced by Dirac [5], provides a natural framework to include the relativistic effects which are crucial in describing the low-lying mesons. The simple vacuum except the zero-mode in LFD is a remarkable feature and facilitates an effective description of hadrons in a Fock space. In this formalism, each hadron is characterized by a set of Fock state wave functions, i.e. the probability amplitudes for finding different combinations of bare quarks and gluons in the hadron at a given LF time $\tau = t + z/c$. Moreover, the LF wave function is independent of all reference frames related by the front-form boosts because the longitudinal boost operator as well as the LF transverse boost operators are all kinematical. This is clearly an advantageous feature unique to LFD, which makes the calculation of observables such as form factors much more effective than any other forms of relativistic dynamics.

Taking advantage of the LFD, we have developed [6, 7] a light-front quark model (LFQM) based on a simple QCD-motivated effective Hamiltonian for a description of mesons. The key idea of our LFQM was to treat the radial wave function as a trial function for the variational principle to the QCD-motivated effective Hamiltonian with the well-known linear plus Coulomb interaction. Both the masses and the hadronic properties of ground state pseudoscalar and vector mesons in our LFQM were fairly well reproduced by taking just a $1S$-state harmonic oscillator (HO) wave function as a trial function. Computing the meson mass spectra [6–8], however, we have treated the hyperfine interaction as a perturbation rather than including it in the variation procedure to avoid the negative infinity from the Dirac delta function contained in the hyperfine interaction. This treatment, however, left a room for any further development in handling the hyperfine interaction in our LFQM since the justification of using the perturbation was too technical to be backed up by the first principle, although our numerical results were quite comparable with the available experimental data not only for the meson mass spectra but also for the meson wave function related observables such as the decay constants, form factors, etc.

The main purpose of this work is to update our previous model [6–8] by including the hyperfine interaction also in our variational method to get the optimal model parameters, and examine if it improves our numerical results compared to the ones obtained by the perturbative treatment of the hyperfine interaction. To achieve this goal, we smeared out the Dirac delta function by a Gaussian distribution in order to resolve the infinity problem when variational principle is applied. Moreover, we improved our trial wave function by taking a larger HO ba-

sis to analyze this effect of improving trial wave functions on our numerical results. To examine our updated model prediction, we calculate both the meson mass spectra and the meson decay constants.

The paper is organized as follows: In Sec. II, we describe our QCD-motivated effective Hamiltonian with the smeared-out hyperfine interaction. Using two different radial wave functions, i.e. the ground state HO wave function and the mixture of the two lowest order HO states, as trial wave functions of the variational principle, we find the analytic formulae of the mass eigenvalues for the ground state pseudoscalar and vector mesons. The optimum values of model parameters are also presented in this section. In Sec. III, we present our numerical results of the mass spectra obtained from both trial wave functions and compare them with the experimental data as well as our previous calculations [6–8]. To test our trial wave functions with the parameters obtained from the variational principle, we also calculate the meson decay constants and compare them with the experimental data as well as other available theoretical predictions. We show an improvement of our numerical results by taking a larger HO basis in the trial wave function over the case of taking just the ground state HO wave function as the trial wave function. Summary and conclusion follow in Sec. IV. The detailed procedure of fixing our parameters through variational principle is presented in the Appendix A.

II. MODEL DESCRIPTION

In our LFQM for mesons, we approximate the system as effectively dressed valence quarks governed by the following QCD-motivated effective Hamiltonian in which motion of the quarks inside a meson is relativistic [6–8]:

$$H = \sqrt{m_q^2 + \vec{k}^2} + \sqrt{m_{\bar{q}}^2 + \vec{k}^2} + V, \tag{1}$$

where $\vec{k} = (\mathbf{k}_\perp, k_z)$ is the relativistic three-momentum of the constituent quarks in the rest frame of the meson. To describe the interaction between quark and antiquark, we use the linear confining potential $V_{\rm conf}$ plus the one-gluon exchange potential $V_{\rm oge}$. For S-wave pseudoscalar and vector mesons, the one-gluon exchange potential reduces to the coulomb potential $V_{\rm coul}$ plus the hyperfine interaction $V_{\rm hyp}$. Thus, we have

$$\begin{aligned} V &= V_{\rm conf} + V_{\rm oge} \\ &= \underbrace{a\ +\ b\,r}_{\rm conf}\ \underbrace{\overbrace{-\ \frac{4\alpha_s}{3r}}^{\rm coul}\ +\ \overbrace{\frac{2}{3}\frac{\mathbf{S}_q\cdot\mathbf{S}_{\bar{q}}}{m_q m_{\bar{q}}}\nabla^2 V_{\rm coul}}^{\rm hyp}}_{\rm oge}, \end{aligned} \tag{2}$$

where $\langle \mathbf{S}_q \cdot \mathbf{S}_{\bar{q}} \rangle = 1/4\ (-3/4)$ for the vector (pseudoscalar) meson and $\nabla^2 V_{\rm coul} = \dfrac{16\pi\alpha_s}{3}\delta^3(\mathbf{r})$. The LF wave function of the ground state mesons is given by

$$\Psi^{JJ_z}_{100}(x_i, \mathbf{k}_{\perp i}, \lambda_i) = \mathcal{R}^{JJ_z}_{\lambda_q\lambda_{\bar{q}}}(x_i, \mathbf{k}_{\perp i})\phi(x_i, \mathbf{k}_{\perp i}), \tag{3}$$

where ϕ is the radial wave function and $\mathcal{R}^{JJ_z}_{\lambda_q\lambda_{\bar{q}}}$ is the interaction-independent spin-orbit wave function. The wave function is represented by the Lorentz invariant internal variables $x_i = p_i^+/P^+$, $\mathbf{k}_{\perp i} = \mathbf{p}_{\perp i} - x_i \mathbf{P}_\perp$ and helicity λ_i, where $P^\mu = (P^+, P^-, \mathbf{P}_\perp) = (P^0 + P^3, (M^2 + \mathbf{P}_\perp^2)/P^+, \mathbf{P}_\perp)$ is the momentum of the meson with mass M and p_i^μ is the momenta of constituent quarks.

The spin-orbit wave functions for pseudoscalar and vector mesons are given by [8, 9]

$$\begin{aligned} \mathcal{R}^{00}_{\lambda_q\lambda_{\bar{q}}} &= \frac{-\bar{u}_{\lambda_q}(p_q)\gamma_5 \nu_{\lambda_{\bar{q}}}(p_{\bar{q}})}{\sqrt{2}\tilde{M}_0}, \\ \mathcal{R}^{1J_z}_{\lambda_q\lambda_{\bar{q}}} &= \frac{-\bar{u}_{\lambda_q}(p_q)\left[\not{\epsilon}(J_z) - \frac{\epsilon\cdot(p_q - p_{\bar{q}})}{M_0 + m_q + m_{\bar{q}}}\right]\nu_{\lambda_{\bar{q}}}(p_{\bar{q}})}{\sqrt{2}\tilde{M}_0}, \end{aligned} \tag{4}$$

where $\epsilon^\mu(J_z)$ is the polarization vector of the vector meson and $\tilde{M}_0 = \sqrt{M_0^2 - (m_q - m_{\bar{q}})^2}$. The boost invariant meson mass squared M_0^2 obtained from the free energies of the constituents is given by

$$M_0^2 = \frac{\mathbf{k}_\perp^2 + m_q^2}{x} + \frac{\mathbf{k}_\perp^2 + m_{\bar{q}}^2}{1-x}. \tag{5}$$

The spin-orbit wave functions satisfy the relation $\sum_{\lambda_q\lambda_{\bar{q}}} \mathcal{R}^{JJ_z\dagger}_{\lambda_q\lambda_{\bar{q}}}\mathcal{R}^{JJ_z}_{\lambda_q\lambda_{\bar{q}}} = 1$ for both pseudoscalar and vector mesons. So the computation on this part of the wavefunction is rather trivial in our approach. We will thus mainly focus on the radial wave functions in the following.

To use a variational principle, we take our trial wave function as an expansion of the true wave function in the HO basis. We use the same trial wave function ϕ for both pseudoscalar and vector mesons, but we try two different forms: one simply takes the form of the $1S$-state HO wave function given by

$$\phi_{1S}(x_i, \mathbf{k}_{\perp i}) = \frac{4\pi^{3/4}}{\beta^{3/2}}\sqrt{\frac{\partial k_z}{\partial x}}\exp(-\vec{k}^2/2\beta^2), \tag{6a}$$

and the other one is expanded with the two lowest order HO wave functions

$$\sqrt{f}\ \phi_{1S}(x_i, \mathbf{k}_{\perp i}) + \sqrt{1-f}\ \phi_{2S}(x_i, \mathbf{k}_{\perp i}), \tag{6b}$$

where f gives the fraction of $1S$ wave function and

$$\begin{aligned} &\phi_{2S}(x_i, \mathbf{k}_{\perp i}) \\ &= \frac{4\pi^{3/4}}{\sqrt{6}\beta^{7/2}}\left(2\vec{k}^2 - 3\beta^2\right)\sqrt{\frac{\partial k_z}{\partial x}}\exp(-\vec{k}^2/2\beta^2). \end{aligned} \tag{7}$$

For convenience, we will call Eq. (6a) and Eq. (6b) our wave function $\phi_{\rm A}$ and $\phi_{\rm B}$, respectively. In these equations, β is the variational parameter and k_z is the longitudinal momentum defined by $k_z = (x-1/2)M_0 + (m_{\bar{q}}^2 - m_q^2)/2M_0$. Thus the Jacobian of the variable transformation $(x, \mathbf{k}_\perp) \to \vec{k} = (\mathbf{k}_\perp, k_z)$ is given by

$$\frac{\partial k_z}{\partial x} = \frac{M_0}{4x(1-x)}\left\{1 - \left[\frac{m_q^2 - m_{\bar{q}}^2}{M_0^2}\right]^2\right\}. \tag{8}$$

The normalization factor in ϕ_{nS} is obtained from the following normalization of the wave function:

$$\int_0^1 dx \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}|\phi_{nS}(x_i, \mathbf{k}_{\perp i})|^2 = 1. \tag{9}$$

Next, we evaluate the expectation value of the Hamiltonian in Eq. (1) with $\phi_{\rm A(B)}$, i.e. $\langle\phi_{\rm A(B)}|H|\phi_{\rm A(B)}\rangle$ which depends on the variational parameter β. According to the variational principle, we can set the upper limit of the ground state's energy by calculating the expectation value of the system's Hamiltonian with a trial wave function. In our previous calculations [6–8], which we call "CJ model", we first evaluate the expectation value of the central Hamiltonian $T + V_{\rm conf} + V_{\rm coul}$ with the trial function $\phi_{\rm A}$, where T is the kinetic energy part of the Hamiltonian. Once the model parameters are fixed by minimizing the expectation value $\langle\phi_{\rm A}|(T + V_{\rm conf} + V_{\rm coul})|\phi_{\rm A}\rangle$, then the mass eigenvalue of each meson is obtained as $M_{q\bar{q}} = \langle\phi_{\rm A}|H|\phi_{\rm A}\rangle$. The hyperfine interaction $V_{\rm hyp}$ in CJ model, which contains a Dirac delta function, was treated as perturbation to the Hamiltonian and was left out in the variational process that optimizes the model parameters. The main reason for doing this was to avoid the negative infinity generated by the delta function as was pointed out in [10]. Specifically, $\langle\phi_{\rm A}|V_{\rm hyp}|\phi_{\rm A}\rangle$ for pseudoscalar mesons decreases faster than other terms that increase as β increases. So the expectation value of the Hamiltonian is unbounded from below.

But now we want to include the hyperfine interaction in our parameterization process. To avoid the negative infinity, we use a Gaussian smearing function to weaken the singularity of $\delta^3(\mathbf{r})$ in hyperfine interaction, viz. [10, 11],

$$\delta^3(\mathbf{r}) \to \frac{\sigma^3}{\pi^{3/2}} e^{-\sigma^2\mathbf{r}^2}. \tag{10}$$

Once the delta function is smeared out as in Eq. (10), a true minimum for the mass occurs at a finite value of β. The analytic formulae of mass eigenvalues for our modified Hamiltonian with the smeared-out hyperfine interaction, i.e. $M_{q\bar{q}}^{\rm A(B)} = \langle\phi_{\rm A(B)}|H|\phi_{\rm A(B)}\rangle$, are found as follows

$$\begin{aligned} M_{q\bar{q}}^{\rm A} = a + \frac{2b}{\sqrt{\pi}\beta} + \frac{1}{\sqrt{\pi}\beta}\sum_{i=q,\bar{q}} m_i^2 e^{m_i^2/2\beta^2} K_1\left(\frac{m_i^2}{2\beta^2}\right) \\ - \alpha_s\left[\frac{8\beta}{3\sqrt{\pi}} - \frac{32\langle\mathbf{S}_q\cdot\mathbf{S}_{\bar{q}}\rangle\beta^3\sigma^3}{9m_q m_{\bar{q}}\sqrt{\pi}\left(\beta^2+\sigma^2\right)^{3/2}}\right], \end{aligned} \tag{11}$$

and

$$\begin{aligned} M_{q\bar{q}}^{\rm B} =& a + b\left(\frac{3}{\sqrt{\pi}\beta} - \frac{f}{\sqrt{\pi}\beta} - 2\sqrt{\frac{2}{3\pi}}\frac{\sqrt{(1-f)f}}{\beta}\right) \\ &+ \frac{1}{6\sqrt{\pi}\beta^5}\left\{6\sqrt{\pi}\beta^6\left(3f + \sqrt{6}\sqrt{(1-f)f} - 3\right)\left[U\left(-\frac{1}{2}, -2, \frac{m_{\bar{q}}^2}{\beta^2}\right) + U\left(-\frac{1}{2}, -2, \frac{m_q^2}{\beta^2}\right)\right]\right. \\ &+ 2(f-1)m_{\bar{q}}^4 e^{\frac{m_{\bar{q}}^2}{2\beta^2}}\left(m_{\bar{q}}^2 - 3\beta^2\right)K_2\left(\frac{m_{\bar{q}}^2}{2\beta^2}\right) - m_{\bar{q}}^2 e^{\frac{m_{\bar{q}}^2}{2\beta^2}}\left[2(f-1)m_{\bar{q}}^4 + 3\beta^4\left(f + 2\sqrt{6}\sqrt{(1-f)f} - 3\right)\right]K_1\left(\frac{m_{\bar{q}}^2}{2\beta^2}\right) \\ &\left.+2(f-1)m_q^4 e^{\frac{m_q^2}{2\beta^2}}\left(m_q^2 - 3\beta^2\right)K_2\left(\frac{m_q^2}{2\beta^2}\right) - m_q^2 e^{\frac{m_q^2}{2\beta^2}}\left[3\beta^4\left(f + 2\sqrt{6}\sqrt{(1-f)f} - 3\right) + 2(f-1)m_q^4\right]K_1\left(\frac{m_q^2}{2\beta^2}\right)\right\} \\ &+ \alpha_s\left\{\frac{16\beta^3\sigma^3\langle\mathbf{S}_q\cdot\mathbf{S}_{\bar{q}}\rangle\left[2\beta^4 + 2\beta^2\left(2f + \sqrt{6}\sqrt{(1-f)f}\right)\sigma^2 + \left(-f + 2\sqrt{6}\sqrt{(1-f)f} + 3\right)\sigma^4\right]}{9\sqrt{\pi}\left(\beta^2+\sigma^2\right)^{7/2} m_q m_{\bar{q}}}\right. \\ &\left.-\frac{20\beta}{9\sqrt{\pi}} - \frac{4\beta f}{9\sqrt{\pi}} - \frac{8}{3}\sqrt{\frac{2}{3\pi}}\beta\sqrt{(1-f)f}\right\}, \end{aligned} \tag{12}$$

where K_1 is the modified Bessel function of the second kind and $U(a,b,z)$ is Tricomi's (confluent hypergeomet-

ric) function.

We then apply the variational principle, i.e. $\partial M_{q\bar{q}}^{\mathrm{A(B)}}/\partial\beta = 0$, to find the optimal model parameters in order to get a best fit for the mass spectra of ground state pseudoscalar and vector mesons (a more detailed description of this procedure can be found in Appendix A).

Our optimized potential parameters are obtained as $\{a = -0.5575$ GeV, $b = 0.18$ GeV^2, and $\alpha_s = 0.5174\}$ for $M_{q\bar{q}}^{\mathrm{A}}$ and $\{a = -0.6664$ GeV$, b = 0.18$ GeV$^2, \alpha_s = 0.5348\}$ for $M_{q\bar{q}}^{\mathrm{B}}$, respectively. We should note that the two sets of potential parameters are quite comparable with the ones suggested by Scora and Isgur [12], where they obtained $a = -0.81$ GeV, $b = 0.18$ GeV^2, and $\alpha_s = 0.3 \sim 0.6$. For a comparison, the coupling constant we found in our previous model [6–8] was $\alpha_s = 0.31$. For the best fit of the mass spectra, the mixing factor f for ϕ_{B} is found to be 0.7, i.e. 70% of the $1S$ state contribution to ϕ_{B}.

Since we included the hyperfine interaction with smearing function entirely in our variational process, we now obtained the two different sets of β values, one for pseudoscalar and the other for vector mesons, respectively. The optimal Gaussian parameters $\beta_{q\bar{q}}$ for pseudoscalar and vector mesons are listed in Table I and II, respectively. Our optimal constituent quark masses and the smearing parameter σ are also listed in Table III.

Our modified model with the smeared-out hyperfine interaction improves the mass spectrum fitting, which is presented in the next section. This may suggest that when using constituent quark models, the contact interactions has to be smeared out. In fact, we think this smeared-out interaction seems to be more consistent with the physical picture for a system of finite-sized constituent quarks.

For practical application of our model, we also compute the decay constants for the ground state pseudoscalar and vector mesons. The decay constants are defined by

$$\langle 0|\bar{q}\gamma^\mu\gamma_5 q|P\rangle = i f_P P^\mu,$$
$$\langle 0|\bar{q}\gamma^\mu q|V(P,h)\rangle = f_V M_V \epsilon^\mu(h), \quad (13)$$

for pseudoscalar and vector mesons, respectively. The experimental values of the pion and rho meson decay constants are $f_\pi \approx 131$ MeV from $\pi \to \mu\nu$ and $f_\rho \approx 220$ MeV from $\rho \to e^+e^-$.

Using the plus component ($\mu = +$) of the currents, one can easily calculate the decay constants. The explicit formulae of pseudoscalar and vector meson decay constants are given by [8, 9]

$$f_P = 2\sqrt{6}\int \frac{dx d^2\mathbf{k}_\perp}{16\pi^3} \frac{\mathcal{A}}{\sqrt{\mathcal{A}^2 + \mathbf{k}_\perp^2}} \phi(x, \mathbf{k}_\perp),$$
$$f_V = 2\sqrt{6}\int \frac{dx d^2\mathbf{k}_\perp}{16\pi^3} \frac{\phi}{\sqrt{\mathcal{A}^2 + \mathbf{k}_\perp^2}} \left[\mathcal{A} + \frac{2\mathbf{k}_\perp^2}{\mathcal{M}_0}\right] \quad (14)$$

where $\mathcal{A} = (1-x)m_q + x m_{\bar{q}}$ and $\mathcal{M}_0 = M_0 + m_q + m_{\bar{q}}$.

FIG. 1. (color online). Fit of the ground state meson masses [MeV] with the parameters given in Table I, II and III for ϕ_{A} and ϕ_{B}, compared with the fit from our previous calculations using CJ model [8] as well as the experimental values. The (π, ρ) masses are our input data.

Here only the $L_z = S_z = 0$ component of the wave function contributes. Note that the vector meson decay constant f_V is extracted from the longitudinal ($h = 0$) polarization. We perform the decay constant calculations for both trial wave functions ϕ_{A} and ϕ_{B} using the corresponding set of parameters fixed for each trial wavefunction, respectively.

III. RESULTS AND DISCUSSION

We show in Fig. 1 our prediction of the meson mass spectra obtained from the variational principle to the modified Hamiltonian with the smeared-out hyperfine interaction using two different trial functions ϕ_{A} (blue lines) and ϕ_{B} (purple lines) and compare them with the experimental data (green lines). We also include the results (black lines) obtained from the CJ model with the linear confining potential [8]. We should note that the masses of π and ρ mesons are used as inputs in our calculation. As one can see, the single $1S$ state HO

TABLE I. The Gaussian parameter β [GeV] for ground state pseudoscalar mesons obtained by the variational principle. q = u and d.

Model	$\beta_{\rm qq}$	$\beta_{\rm qs}$	$\beta_{\rm qc}$	$\beta_{\rm cs}$	$\beta_{\rm cc}$	$\beta_{\rm qb}$	$\beta_{\rm bs}$	$\beta_{\rm bc}$	$\beta_{\rm bb}$
$\phi_{\rm A}$	0.6376	0.5513	0.5810	0.5994	0.7916	0.6686	0.7132	1.0577	1.6455
$\phi_{\rm B}$	0.4520	0.3799	0.3960	0.4078	0.5286	0.4461	0.4757	0.6891	1.0549

TABLE II. The Gaussian parameter β [GeV] for ground state vector mesons obtained by the variational principle. q = u and d.

Model	$\beta_{\rm qq}$	$\beta_{\rm qs}$	$\beta_{\rm qc}$	$\beta_{\rm cs}$	$\beta_{\rm cc}$	$\beta_{\rm qb}$	$\beta_{\rm bs}$	$\beta_{\rm bc}$	$\beta_{\rm bb}$
$\phi_{\rm A}$	0.3480	0.3952	0.5283	0.5727	0.7849	0.6436	0.7010	1.0554	1.6450
$\phi_{\rm B}$	0.2416	0.2742	0.3579	0.3892	0.5233	0.4278	0.4671	0.6871	1.0544

TABLE III. Constituent quark masses [GeV] and the smearing parameter σ [GeV] obtained by the variational principle for the Hamiltonian with a smeared-out hyperfine interaction. Here q = u and d.

Model	$m_{\rm q}$	$m_{\rm s}$	$m_{\rm c}$	$m_{\rm b}$	σ
$\phi_{\rm A}$	0.220	0.432	1.77	5.2	0.405
$\phi_{\rm B}$	0.221	0.456	1.77	5.2	0.423

wave function $\phi_{\rm A}$ already generates a good enough fitting for the spectrum, and a more complicated trial wave function $\phi_{\rm B}$ does not change the $1S$ results too much. In fact, the χ^2 value for this modified model is 0.014 [0.018] for $\phi_{\rm A}[\phi_{\rm B}]$, which is more than half reduced from $\chi^2 = 0.039$ for the CJ model [8]. Except for the mass of K, our predictions for the masses of $1S$-state pseudoscalar and vector mesons are within 4% error. Especially, our modified Hamiltonian clearly improves the predictions of heavy-light and heavy quarkonia systems such as $(\eta_c, J/\psi, B_c, \eta_b, \Upsilon)$ compared to the CJ model adopting the contact hyperfine interaction. Although the experimental data for B_c^* is not yet available, our predictions of B_c^*, i.e. 6343 (6325) MeV for $\phi_{\rm A(B)}$, are quite comparable with other quark model predictions such as 6345.8 MeV [13] and 6340 MeV [11].

In Table IV, we list our predictions for the decay constants of light mesons (π, K, ρ, K^*) obtained by using $\phi_{\rm A(B)}$ and compare them with CJ model [14] and the experimental data [15, 16]. As one can see, our updated model calculation including the hyperfine interaction in the variation procedure doesn't seem to improve the results of CJ model. In particular, the trial wave function $\phi_{\rm A}$ generates decay constants that are quite high for light mesons (π, ρ, K, K^*) indicating that just $1S$-state HO wave function alone cannot be a good trial wave function for the entire Hamiltonian including the smeared hyperfine interaction. However, using the improved trial wave function $\phi_{\rm B}$, we can see a dramatic decrease in the numerical results consistent with the variational principle. Indeed, the results from $\phi_{\rm B}$ are much closer to the experimental data than those from $\phi_{\rm A}$. Especially for π, the decay constant changes from 155 MeV to 139 MeV, which is much closer to the experimental value. Although the CJ model yields the experimental value of f_π much better than the updated model results from $\phi_{\rm B}$, an overall improvement due to the change of trial wave functions from $\phi_{\rm A}$ to $\phi_{\rm B}$ seems quite clear. Since the experimental values are very well known for light mesons, this improvement is very encouraging.

In Table V, we list our predictions for the charmed meson decay constants $(f_D, f_{D^*}, f_{D_s}, f_{D_s^*}, f_{\eta_c}, f_{J/\Psi})$ together with CJ Model [17], lattice QCD [18, 19], QCD sum rules [20], relativistic Bethe-Salpeter (BS) model [21], relativized quark model [22], and other relativistic quark model (RQM) [23] predictions as well as the available experimental data [15, 16, 24, 25]. We extract the experimental value $(f_{J/\Psi})_{\rm exp} = (416 \pm 6)$ MeV from the data $\Gamma_{\rm exp}(J/\Psi \to e^+e^-) = 5.55 \pm 0.14 \pm 0.02$ keV and the formula

$$\Gamma(V \to e^+e^-) = \frac{4\pi}{3}\frac{\alpha^2}{M_V} f_V^2 c_V, \tag{15}$$

where $c_V = 4/9$ for $V = J/\Psi$. We should note that our results of the ratios $f_{D_s}/f_D = 1.13[1.14]$ and $f_{\eta_c}/f_{J/\Psi} = 0.88[0.91]$ obtained from $\phi_{\rm A}[\phi_{\rm B}]$ are quite comparable with the available experimental data, $f_{D_s}/f_D = 1.25 \pm 0.06$ [15, 16] and $f_{\eta_c}/f_{J/\Psi} = 0.81 \pm 0.19$ [24, 25], respectively. Our results of the ratios $f_{D_s^*}/f_{D^*} = 1.15[1.17]$ obtained from $\phi_{\rm A}[\phi_{\rm B}]$ are also in good agreement with the BS model prediction, $f_{D_s^*}/f_{D^*} = 1.10 \pm 0.06$ [21]

We list our results for the bottomed mesons $(f_B, f_{B^*}, f_{B_s}, f_{B_s^*}, f_{\eta_b}, f_\Upsilon)$ in Table VI, and compare with CJ Model [17], lattice QCD [18, 26, 27], QCD sum rules [20, 28], BS model [21], relativized quark model [22], and RQM [23] predictions as well as the available experimental data [24, 29]. Note that we extract the experimental value $(f_\Upsilon)_{\rm exp} = (715 \pm 5)$ MeV from the data $\Gamma_{\rm exp}(\Upsilon \to e^+e^-) = 1.340 \pm 0.018$ keV [24] and Eq. (15) with $c_V = 1/9$ for $V = \Upsilon$. Our results for the ratios $f_{B_s}/f_B = 1.17[1.18]$, $f_{B_s^*}/f_{B^*} = 1.19[1.20]$ obtained from $\phi_{\rm A}[\phi_{\rm B}]$ are in good agreement with the re-

cent lattice results, 1.20(3)(1) [26] and 1.22 $\left(^{+5}_{-6}\right)$ [27] for f_{B_s}/f_B and $1.17(4)^{+1}_{-3}$ [18] for $f_{B_s^*}/f_{B^*}$. Our results for the ratio $f_{\eta_b}/f_\Upsilon = 0.92[0.95]$ are to be compared with the $f_{\eta_b}/f_\Upsilon \sim 1$ in HQSS limit [30]. One can also see that for heavy charmed and bottomed mesons, the trial wave function $\phi_{\rm B}$ produces better results when compared with the experimental data as well as the the lattice results.

In Table VII, we present our model predictions for the decay constants of f_{B_c} and $f_{B_c^*}$, and compare them with other model calculations [8, 31–36]. Both trial wave functions $\phi_{\rm A}$ and $\phi_{\rm B}$ give results that are comparable with other model calculations.

IV. SUMMARY AND CONCLUSION

In this work, we updated our LFQM by smearing out the Dirac delta function in the hyperfine interaction and including the smeared hyperfine interaction in our calculation based on the variational principle rather than using the perturbation method to handle the delta function in the contact hyperfine interaction. Using the two trial wave functions, i.e. the $1S$ state HO wave function $\phi_{\rm A}$ and the mixed wave function $\phi_{\rm B}$ of $1S$ and $2S$ HO states, we calculated both the mass spectra of the ground state pseudoscalar and vector mesons and the decay constants of the corresponding mesons. We found that our predictions of the meson mass spectra are in good agreement with the data both for $\phi_{\rm A}$ and $\phi_{\rm B}$, which seems to suggest that for the energy eigenvalues, a simple trial wave function like $\phi_{\rm A}$ may suffice. However, the mixed wave function $\phi_{\rm B}$ of $1S$ and $2S$ states turns out to be much better in the calculation of decay constants as the wave function $\phi_{\rm A}$ generates results that are noticeably higher. According to the variational principle, the results from $\phi_{\rm B}$ are obviously closer to the true results compare to the results from $\phi_{\rm A}$. This improvement is especially evident for the light mesons (π, K, ρ, K^*) where the decay constants are very well known.

The fact that by including one more higher order HO basis in the wave function can improve the results for the decay constants significantly is quite encouraging. Since the more realistic wave function for our modified Hamiltonian in Eq. (1) can in principle be written in a complete HO basis, i.e. $\sum_{n=1}^{n_{\max}} c_n\phi_{nS}$, the improvement obtained by $\phi_{\rm B} = \sum_{n=1}^{2} c_n\phi_{nS}$ is consistant with our expectation. Nonetheless, we should point out that, strictly speaking, we are not comparing trial wave functions for exactly the same Hamiltonian here, since we not only varied the parameters in our wave functions, but also adjusted the parameters in the Hamiltonian, such as the masses of quarks and the smearing parameter for best fitting of the mass spectra, as can be seen in Table III. This makes the Hamiltonian we used for $\phi_{\rm A}$ and $\phi_{\rm B}$ slightly different. However, the parameters in the Hamiltonian are kept within a limited range and thus one can still anticipate that a more sophisticated trial wave function should produce better predictions from the same form of Hamiltonian. Our calculation seems to suggest that our modification of the Hamiltonian by smearing out the hyperfine interaction contributes to an improvement in the right direction.

Since our modified Hamiltonian together with the improved trial wave function provides very good results as discussed in this work, it may be also desirable to investigate further and see if we can improve our previous calculations of other wave function related observables such as form factors. We shall explore them in our future work.

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Appendix A: Fixation of the model parameters using variational principle

In our model, we assumed SU(2) symmetry and have the following parameters that need to be fixed: constituent quark masses $(m_{u(d)}, m_s, m_c, m_b)$, potential parameters (a, b, α_s), gaussian parameter β, and the smearing parameter σ. For our second trial wave function $\phi_{\rm B}$, we also have the mixing factor f that we have to adjust. Notice that the β values here are not only different for different quark combinations, but also different for pseudoscalar and vector mesons of the same quark combination. The reason for this is that the hyperfine interaction we included in our parameterization process gives different contributions to the masses of pseudoscalar and vector mesons and thus induces different parameterizations under variational principle.

We now illustrate our procedure for fixing these parameters. The variational principle gives us one constraint:

$$\frac{\partial\langle\Psi|H|\Psi\rangle}{\partial\beta} = \frac{\partial M_{q\bar{q}}}{\partial\beta} = 0. \tag{A1}$$

We can use this equation to rewrite the coupling constant α_s in terms of other parameters and plug it back into Eq. (11) [Eq. (12)] and thus eliminate α_s. The string constant b is fixed to be 0.18 GeV, a well known value from other quark model analysis [11, 12, 37]. We will leave the quark masses and smearing parameter σ (and the mixing factor f for $\phi_{\rm B}$) as externally adjustable variables. We picked a set of values for $(m_{u(d)}, m_s, m_c, m_b, \sigma)$ when using $\phi_{\rm A}$ and $(m_{u(d)}, m_s, m_c, m_b, \sigma, f)$ when using $\phi_{\rm B}$, and proceed with the following procedure to solve for the rest of parameters. We then vary these values to do the procedure again until we find a set of quark masses

TABLE IV. Decay Constants for light mesons (in unit of MeV) obtained from our updated LFQM.

	f_π	f_ρ	f_K	f_{K^*}
Model $\phi_{\rm A}$	155	234	190	261
Model $\phi_{\rm B}$	139	211	176	234
CJ Model [14]	130	246	161	256
Exp. [15, 16]	130.4 ± 0.2	221 ± 1	156.1 ± 0.8	217 ± 7

TABLE V. Charmed meson decay constants (in unit of MeV) obtained from our updated LFQM.

	f_D	f_{D^*}	f_{D_s}	$f_{D_s^*}$	f_{η_c}	$f_{J/\psi}$
Model $\phi_{\rm A}$	244	279	276	322	406	460
Model $\phi_{\rm B}$	218	241	249	282	354	390
CJ Model [17]	197	239	232	273	326	360
Lattice [18]	$211 \pm 3 \pm 17$	$245 \pm 20^{+3}_{-2}$	$231 \pm 12^{+8}_{-1}$	$272 \pm 16^{+3}_{-20}$	–	–
QCD [19]	$201 \pm 3 \pm 17$	–	$249 \pm 3 \pm 16$	–	–	–
Sum-rules [20]	204 ± 20	–	235 ± 24	–	–	–
BS [21]	230 ± 25	340 ± 23	248 ± 27	375 ± 24	292 ± 25	459 ± 28
QM [22]	240 ± 20	–	290 ± 20	–	–	–
RQM [23]	234	310	268	315	–	–
Exp.	206.7 ± 8.9 [15, 16]	–	257.5 ± 6.1 [15, 16]	–	335 ± 75 [25]	416 ± 6 [24]

TABLE VI. Bottomed meson decay constants (in unit of MeV) obtained from our updated LFQM.

	f_B	f_{B^*}	f_{B_s}	$f_{B_s^*}$	f_{η_b}	f_Υ
Model $\phi_{\rm A}$	229	243	267	288	805	871
Model $\phi_{\rm B}$	195	202	229	242	654	692
CJ Model [17]	171	185	205	220	507	529
Lattice [18]	$179 \pm 18^{+34}_{-9}$	$196 \pm 24^{+39}_{-2}$	$204 \pm 16^{+41}_{-0}$	$229 \pm 20^{+41}_{-16}$	–	–
QCD1 [26]	216 ± 22	–	259 ± 32	–	–	–
QCD2 [27]	189 ± 27	–	230 ± 30	–	–	–
Sum-rules1 [28]	210 ± 19	–	244 ± 21	–	–	–
Sum-rules2 [20]	203 ± 23	–	236 ± 30	–	–	–
BS [21]	196 ± 29	238 ± 18	216 ± 32	272 ± 20	–	498 ± 20
QM [22]	155 ± 15	–	210 ± 20	–	–	–
RQM [23]	189	219	218	251	–	–
Exp.	229^{+36+34}_{-31-37} [29]	–	–	–	–	715 ± 5 [24]

TABLE VII. Bottom-charmed meson decay constants(in unit of MeV) obtained from our updated LFQM.

	$\phi_{\rm A}$	$\phi_{\rm B}$	CJ Model [8]	[31]	[32]	[33]	[34]	[35]	[36]
f_{B_c}	488	406	349	360	433	500	460 ± 60	517	410 ± 40
$f_{B_c^*}$	530	432	369	–	503	500	460 ± 60	517	–

that give best fit for the meson mass spectra. In addition, we have 3 more parameters $(a, \beta^p_{q\bar{q}}, \beta^v_{q\bar{q}})$ to be fixed for mesons of a certain quark combination $(q\bar{q})$, where $\beta^p_{q\bar{q}}$, $\beta^v_{q\bar{q}}$ are the gaussian parameters for pseudoscalar (p) and vector (v) mesons, respectively.

Using the masses of π and ρ as our input values for

$M^{\mathrm{A}}_{q\bar{q}}$ in Eq. (11) [$M^{\mathrm{B}}_{q\bar{q}}$ in Eq. (12)], and the condition that our coupling constants α_s are the same for all these ground state pseudoscalar and vector mesons, we can fix the three model parameters $(a, \beta^p_{q\bar{q}}, \beta^v_{q\bar{q}})$ for $q = u$ or d from the following three equations:

$$M_\pi(\beta^p_{q\bar{q}}, a) = 0.140, \tag{A2a}$$

$$M_\rho(\beta^v_{q\bar{q}}, a) = 0.77, \tag{A2b}$$

$$\alpha_s(\beta^p_{q\bar{q}}, a) = \alpha_s(\beta^v_{q\bar{q}}, a). \tag{A2c}$$

Solving these equations not only gives us the remaining parameters $a, \beta^p_{q\bar{q}}, \mathrm{and} \beta^v_{q\bar{q}}$, but also the coupling constant α_s which we assumed to be the same for all the mesons we consider here. We can then solve for the β values of all the other mesons using the known α_s value, by equating the α_s expressions for different mesons that we got from Eq. (A1). We thus fixed all the parameters for all the mesons we consider here.

Through our trial and error type of analysis, we found $m_q = 0.220$ GeV$, m_s = 0.432$ GeV$, m_c = 1.77$ GeV$, m_b = 5.2$ GeV$, \sigma = 0.405$ GeV gives best fit of the meson mass spectrum when using trial wave function ϕ_{A}, while $m_q = 0.221$ GeV$, m_s = 0.456$ GeV$, m_c = 1.77$ GeV$, m_b = 5.2$ GeV$, \sigma = 0.423$ GeV$, f = 0.7$ gives best fit when using trial wave function ϕ_{B}. For these values, our obtained β values are listed in Table I and II.

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